MOTION OF AN INCLINED PLATE UNDER A FREE BOUNDARY

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The problem under consideration is the plane steady motion of an ideal incompressible gravity-free fluid of infinite depth past an inclined plate under a free boundary. The formulation of the problem corresponds closely to the motion of a plate in the case of a large Froude number. The solution is obtained by using the methods of the theory of jets.

1. The flow pattern in the physical $z = x + iy$ plane is shown on Fig. 1.

A flow of speed V_0 is impinging upon the plate. The velocity vector at infinity makes an angle α_1 with the plate. Far upstream, the streamline which passes through the forward stagnation point D is at a distance h from the free surface EF. It is assumed that the point B is the rear stagnation point, i.e. that the Joukowski-Chaplygin condition is satisfied.

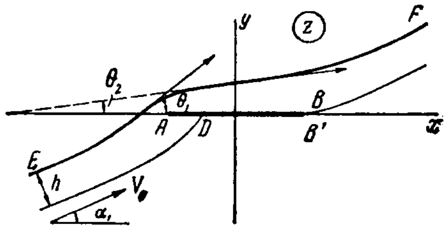

Fig. 1.

Figure 2 shows the situation in terms of the following complex variable:

$$\vartheta = V_0 dz \,/\, dw = (V_0 \,/\, V)\, e^{i\theta}$$

where V is the speed and θ is the angle the velocity vector makes with the real axis. In the ϑ plane, the plate corresponds to the real axis.

The free surface in the ϑ plane corresponds to a circular arc slit on the circle $|\vartheta| = 1$, between the angles of largest inclination θ_1 and smallest inclination θ_2 of the free stream surface. Thus, in the ϑ plane the flow takes place in a doubly connected domain obtained by removing a circular arc from the upper half-plane.

Figure 3 shows the complex potential plane $w = \varphi + i\psi$. It is assumed, (see Fig. 1), that $\psi = 0$ along CD. At the point D we take $\varphi = 0$. Since to one and the same point z on the streamline $\psi = 0$ there may correspond two distinct values of the potential φ, one must have a slit, or cut, along the real axis, $\psi = 0$, to the right of the point D. The difference between the values of φ at the points B and B' (Fig. 3), is obviously equal to the circulation around the plate.

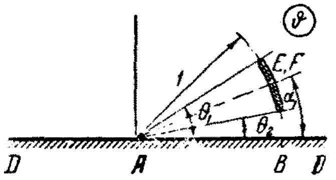

Fig. 2.

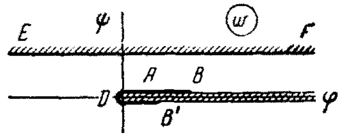

Fig. 3.

The solution of the problem in question may be obtained by determining the conformal representation of the domains of the motion in the ϑ and w planes on a domain in a parametric v plane, that is to find

$$\vartheta = V_0 dz \,/\, dw = f_1(v), \qquad w = f_2(v) \tag{1.1}$$

As the parametric domain we shall choose the interior of a rectangle.

2. Let us map the interior of a rectangle in the v plane conformally onto the domain of the motion in the w plane. First we shall map the w domain on the upper half-plane of an auxiliary variable τ. Let us map the points E, F, D of Fig. 3 into the points $\tau = \infty$, $\tau = 1$ and $\tau = 0$ of the real axis. Employing the Schwarz-Christoffel formula, we obtain

$$w = C_1 [\tau + \ln(\tau - 1) - \pi i] \tag{2.1}$$

For real $\tau \geqslant 1$, the stream function $\psi = hV_0$; taking this into account, and comparing the imaginary parts of both sides of equation (2.1), we can determine C_1 and obtain

$$w = -\frac{hV_0}{\pi} [\tau + \ln(\tau - 1) - \pi i] \tag{2.2}$$

In the τ plane the plate corresponds to the segment $B'B$ of the real axis, and the free surface corresponds to the semi infinite line FE

(Fig. 4).

Let us now map the upper τ half-plane on the interior of the rectangle in the parametric v plane of Fig. 5. Suppose, in doing this, that the segment $B'B$ corresponds to the lower side of the rectangle, while the segment FE corresponds to the upper side of the rectangle.

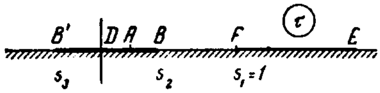

Fig. 4.

Employing anew the Schwarz-Christoffel formula, we may write

$$v = A_1 \int_{\tau}^{\infty} \frac{dt}{\sqrt{(t-s_1)(t-s_2)(t-s_3)}} + A_2 \qquad (2.3)$$

where $s_1 = 1$, s_2 and s_3 are the coordinates of the points F, B and B' in the τ plane. Let us introduce the new variable $t_1 = t + s$; and the abbreviations

$$s_1 + s = e_1', \quad s_2 + s = e_2', \quad s_3 + s = e_3' \qquad (2.4)$$

where s is defined by means of the condition: $e_1' + e_2' + e_3' = 0$.

Then (2.3) may be written as follows:

$$v = 2A_1 \int_{\tau+s}^{\infty} \frac{dt_1}{\sqrt{4(t_1-e_1')(t_1-e_2')(t_1-e_3')}} + A_2$$

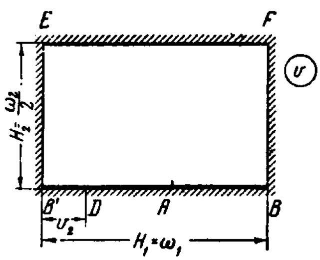

Fig. 5.

Inverting the integral, we obtain

$$\tau + s = \wp_\Omega\left(\frac{v - A_2}{2A_1},\ \Omega_1, \Omega_2\right) \qquad (2.5)$$

where $\wp_\Omega$ is the elliptic function of Weierstrass, with periods Ω_1, Ω_2, which are expressed in terms of e_1', e_2' and e_3' by means of the formulas

$$\Omega_1 = \frac{2K}{\sqrt{e_1'-e_3'}} = \frac{2K}{\sqrt{1-s_3}}, \qquad \Omega_2 = \frac{2K'i}{\sqrt{e_1'-e_3'}} = \frac{2K'i}{\sqrt{1-s_3}} \qquad (2.6)$$

Here K and K' are the complete elliptic integrals of the first kind with the moduli

$$k = \left(\frac{e_2'-e_3'}{e_1'-e_3'}\right)^{1/2} = \left(\frac{s_2-s_3}{1-s_3}\right)^{1/2}, \qquad k' = \sqrt{1-k^2} = \left(\frac{1-s_2}{1-s_3}\right)^{1/2} \qquad (2.7)$$

In turn, the roots e_1', e_2' and e_3' may be expressed in terms of the periods Ω_1, Ω_2 by means of the formulas

$$e_1' = \wp_\Omega(1/2\,\Omega_1), \qquad e_2' = \wp_\Omega(1/2\,(\Omega_1+\Omega_2)), \qquad e_3' = \wp_\Omega(1/2\,\Omega_2) \qquad (2.8)$$

Employing (2.8), and fulfilling the conditions for correspondence between points, we obtain, after simple transformations, that

$$\tau + s = \wp_\Omega (v + {}^1/_2\, \Omega_2) \tag{2.9}$$

The base, H_1, and the height, H_2, of the rectangle, are given by

$$H_1 = {}^1/_2\, \Omega_1, \qquad H_2 = {}^1/_2\, \Omega_2 \tag{2.10}$$

The quantity $2A_1$ has thus been set equal to unity, which amounts to choosing the scale of measurement. From the theory of elliptic functions, [1] we use the known formulas relating $\wp_\Omega(v + 1/2\ \Omega_2)$ and $\wp_\Omega(v)$, and $\wp_\Omega$ and the Jacobi functions, we may write

$$\tau = s_2 sn^2 \left(v \sqrt{1 - s_3}\right) + s_3\, cn^2 \left(v \sqrt{1 - s_3}\right) \tag{2.11}$$

where the functions sn and cn have the modulus k of (2.7).

3. Let us map the domain of the flow in the ϑ plane on the ϑ_1 plane, in such a way that the contour of the "plate" in the ϑ plane goes into the unit circle in the ϑ_1 plane and that the slit corresponding to the free surface, in the ϑ plane is mapped into a portion of the real axis in the ϑ_1 plane.

This required transformation (Fig. 6) is given by

$$\vartheta_1 = -\, i\, \frac{\vartheta + i}{\vartheta - i} \tag{3.1}$$

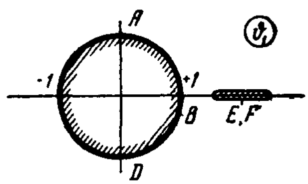

Fig. 6.

Applying to the ϑ_1 plane the Joukowski transformation:

$$\vartheta_2 = {}^1/_2\, (\vartheta_1 + \vartheta_1^{-1}) \tag{3.2}$$

we may transform the unit circle in the ϑ_1 plane on the slit from -1 to $+1$ along the real axis in the ϑ_2 plane. The slit corresponding to the "free surface" also falls on the real ϑ_2 axis. The end points a_1 and a_2 of this slit (Fig. 7) are related to the limiting values of the slope of the free surface by means of the equations

$$a_1 = \frac{1}{\cos \theta_1}, \qquad a_2 = \frac{1}{\cos \theta_2} \tag{3.3}$$

The point at infinity, upstream, on the free surface corresponds to the point on the lower side of the slit, with affix

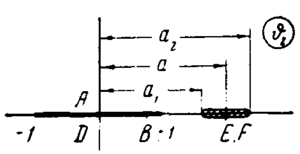

Fig. 7.

$$a = \frac{1}{\cos \alpha_1} \tag{3.4}$$

The exterior of two slits (see [2]) may be mapped onto a rectangle in the u plane

(Fig. 8) by means of the relations

$$\vartheta_2 = \mathrm{im}\,[\zeta\,(u-\alpha) - \zeta\,(u+\alpha)] + C \tag{3.5}$$

where ζ is the Weierstrass function. The "plate" segment goes into the base of the rectangle, of length ω_1, the segment representing the "free surface" goes over into the top side of the rectangle, and the remainder of the real axis goes into the vertical sides, of height $1/2\ \omega_2$. The purely imaginary number α is the image in the u plane of the point at infinity in the ϑ_2 plane. The point at the rear of the plate goes over into a point on the base of the rectangle with affix μ. The constants a, m, C, and the ratio of the periods ω_2/ω_1 determine the coordinates of the end points of the slits in the ϑ_2 plane.

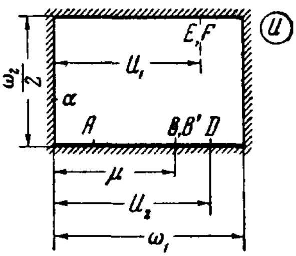

Fig. 8.

Employing a known formula from the theory of elliptic functions, (3.5 may be replaced by

$$\vartheta_2 = \mathrm{im}\left[\frac{\wp'(\alpha)}{\wp(u) - \wp(\alpha)} - 2\zeta\,(\alpha)\right] + C \tag{3.6}$$

where $\wp(u)$ is the elliptic function of Weierstrass, with periods ω_1 and ω_2. On the imaginary axis $\wp(u)$ is real and negative, and $\wp'(u)$ is imaginary and negative. On the real axis $\wp(u)$ varies from ∞ to $\wp(1/2\ \omega_1) = e_1$ and is symmetric with respect to $u = 1/2\ \omega_1$.

It may be shown that, along the base of the rectangle, the value $u = 0$ corresponds to the minimum $\vartheta_2 = -1$, and the value $u = 1/2\ \omega_1$ corresponds to the maximum $\vartheta_2 = +1$. Employing this fact, one may determine the two constants in (3.6); and, after inserting the obtained values of the constants, one may rewrite (3.6) in the form

$$\vartheta_2 = \frac{\wp_1(\alpha) + \wp_1(u)}{\wp_1(\alpha) - \wp_1(u)}, \qquad \wp_1(u) = \wp(u) - e_1 \tag{3.7}$$

The two constants α and ω_2/ω_1 remain free, because the end points of the "free surface" are not prescribed in the ϑ_2 plane.

Let us choose the sides of our rectangle in the u plane equal to the corresponding sides of the rectangle in the v plane. Then, from (2.10), we obtain

$$\omega_1 = {}^1/_2\,\Omega_1, \qquad \omega_2 = \Omega_2 \tag{3.8}$$

If the abscissa u_1 of the points E, F equals the abscissa μ of the points $B'B$, then the conformal mapping of the ϑ_2 domain on the rectangle in the v plane (Fig. 5) may be obtained, in view of the periodicity of the function $\wp(u)$, by putting $u = v + \mu$ in (3.7).

Indeed, if this choice is made, then the contours of the rectangles coincide, the bases correspond to the "plate" and the tops correspond to the "free surface". The positions of the points B', B, E and F will then agree. Let us write down the condition which guarantees that $u_1 = \mu$. The number u_1 may be obtained from (3.7) for $\vartheta_2 = 1/\cos\alpha_1$, according to (3.4).

Carrying out the corresponding calculations for u_1, we obtain the equation

$$\wp(u_1) = e_2\left(\frac{e_3}{e_2}\cdot\frac{\wp_1(\alpha)}{e_1 - e_3}\cdot\frac{\cos\alpha_1 - 1}{\cos\alpha_1 + 1} - 1\right)\left(\frac{\wp_1(\alpha)}{e_1 - e_3}\cdot\frac{\cos\alpha_1 - 1}{\cos\alpha_1 + 1} - 1\right)^{-1} \tag{3.9}$$

where e_1, e_2 and e_3 are connected with the periods ω_1 and ω_2 of the function $\wp(u)$ by relations which are analogous to (2.8).

The number μ may be expressed in terms of the coordinates u_1 and u_2 of the stagnation point D, which corresponds to $\vartheta_2 = 0$ and to $\tau = 0$. From (3.7) and (2.11) we obtain the following equations for the determination of u_2 and v_2:

$$\wp_1(u_2) = -\wp_1(\alpha), \qquad \frac{\operatorname{sn}^2(v_2\sqrt{1 - s_3})}{\operatorname{cn}^2(v_2\sqrt{1 - s_3})} = -\frac{s_3}{s_2} \tag{3.10}$$

From the equation $u = v + \mu$ (Figs. 5 and 8) it follows that one has $\mu = u_2 - v_2$.

The equation $u_1 = \mu$ implies that $u_1 = u_2 - v_2$.

Substituting this result into (3.9), and replacing $\wp(\alpha)$ by means of (3.10), we obtain the following equation for u_2:

$$\wp(u_2 - v_2) = e_2\left(\frac{e_3}{e_2}\,\frac{\wp_1(u_2)}{e_1 - e_3}\,\frac{\cos\alpha_1 - 1}{\cos\alpha_1 + 1} + 1\right)\left(\frac{\wp_1(u_2)}{e_1 - e_3}\,\frac{\cos\alpha_1 - 1}{\cos\alpha_1 + 1} + 1\right)^{-1} \tag{3.11}$$

Employing the addition theorem for $\wp$, and the relation connecting $\wp'$ and $\wp$, we obtain from (3.11) an algebraic equation for $\wp(u_2)$, whose solution yields u_2, after which α may be obtained from (3.10).

In order to determine the constants corresponding to a given angle of attack α, it is convenient to fix first the values s_2 and s_3; then to obtain Ω_1 and Ω_2 from (2.6) and (2.7), then v_2 from (3.10), and then ω_1 and ω_2 from (3.8). Knowing ω_1 and ω_2, one may then compute e_1, e_2 and e_3, by means of formulas which are analogous to (2.8); and then (3.11) may be solved. Further, (3.10) then determined $\wp(\alpha)$, and hence μ may be obtained.

Combining equations (3.1), (3.2) and (3.7), and observing that $u = v + \mu$, we get

$$\frac{2\vartheta}{\vartheta^2 + 1} = \frac{\wp_1(\alpha) + \wp_1(v + \mu)}{\wp_1(\alpha) - \wp_1(v + \mu)} \tag{3.12}$$

by means of which the domain in the ϑ plane is mapped conformally on the rectangle in the parametric plane v. The conformal mapping between the w and v planes is given by the formulas (2.2) and (2.11).

Once $V_0 dz/dw = f_1(v)$ and $w = f_2(v)$ are known, then the pressure distribution on the plate, the resultant forces, the streamlines, and so forth, may be determined by the usual methods [3].

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